

An infinite family of tight triangulated manifolds

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Abstract

We give an explicit construction of vertex-transitive tight triangulated d -manifolds for $d \geq 2$. More explicitly, for each $d \geq 2$, we construct two $(d^2 + 5d + 5)$ -vertex neighborly triangulated d -manifolds whose vertex-links are stacked spheres. The only other non-trivial series of such tight triangulated manifolds currently known is the series of non-simply connected triangulated d -manifolds with $2d + 3$ vertices constructed by Kühnel. The manifolds we construct are strongly minimal. For $d \geq 3$, they are also tight neighborly as defined by Lutz, Sulanke and Swartz. Like Kühnel's complexes, our manifolds are orientable in even dimensions and non-orientable in odd dimensions.

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1 Introduction

In [17], Walkup introduced the class $\mathcal{K}(d)$, $d \geq 2$, of triangulated d -manifolds whose vertex-links are stacked $(d - 1)$ -spheres. For $d = 2$, Walkup's class is the set of all triangulated d -manifolds. The following result by Kalai [10] shows that the members of this class triangulate a very natural class of manifolds obtained by handle additions on a sphere.

Proposition 1.1 (Kalai). *For $d \geq 4$, a connected simplicial complex X is in $\mathcal{K}(d)$ if and only if X is obtained from a stacked d -sphere by $\beta_1(X)$ combinatorial handle additions. In consequence, any such X triangulates either $(S^{d-1} \times S^1)^{\# \beta_1}$ or $(S^{d-1} \times S^1)^{\# \beta_1}$ according as X is orientable or not. (Here $\beta_1 = \beta_1(X) = \beta_1(X; \mathbb{Z}_2)$.)*

Walkup's class $\mathcal{K}(d)$ has also been a major source of examples of tight triangulations. Recall that, for a field \mathbb{F} , a d -dimensional simplicial complex X is called *tight with respect to \mathbb{F}* (or \mathbb{F} -tight) if (i) X is connected, and (ii) for all induced sub-complexes Y of X and for all $0 \leq j \leq d$, the morphism $H_j(Y; \mathbb{F}) \rightarrow H_j(X; \mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ is injective [12, 3]. In this paper, by tight we mean tight with respect to the field \mathbb{Z}_2 .

Very few examples of tight triangulations are known. Apart from the trivial $(d + 2)$ -vertex triangulation S_{d+2}^d of the d -sphere S^d , the only non-trivial series of such triangulations currently known is the $(2d + 3)$ -vertex non-simply connected triangulated manifolds K_{2d+3}^d constructed by Kühnel [11]. The complex K_{2d+3}^d triangulates an S^{d-1} -bundle over S^1 and, for $d \geq 3$ it is the unique non-simply connected triangulated d -manifold with $2d + 3$ vertices [1, 5]. Not surprisingly, Kühnel's triangulations are members of $\mathcal{K}(d)$. Walkup's class also relates to one of the few combinatorial criteria for tightness that are known (for more general combinatorial criteria see [3, Theorem 3.10]). For example, Effenberger [8] and Bagchi and Datta [3] showed that:

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Proposition 1.2 (Effenberger). *For $d \neq 3$, the neighborly members of $\mathcal{K}(d)$ are tight.*

Proposition 1.3 (Bagchi and Datta). *Let M be a neighborly member of $\mathcal{K}(3)$. Then M is tight if and only if $\beta_1(M; \mathbb{Z}_2) = (f_0(M) - 4)(f_0(M) - 5)/20$.*

Walkup's class is also closely related to the notion of tight neighborly triangulation as introduced by Lutz, Sulanke and Swartz in [13]. Indeed they prove:

Proposition 1.4. *Let X be a connected triangulated d -manifold. Then X satisfies*

$$\binom{f_0 - d - 1}{2} \geq \binom{d + 2}{2} \beta_1(X; \mathbb{Z}_2). \quad (1)$$

Moreover for $d \geq 4$, the equality holds if and only if X is a neighborly member of $\mathcal{K}(d)$.

For $d \geq 3$, a triangulated d -manifold is called *tight neighborly* if it satisfies (1) with equality.

In this paper, we present the second infinite series of neighborly members of $\mathcal{K}(d)$ after Kühnel's series K_{2d+3}^d . Like Kühnel's complexes, our manifolds also exhibit vertex-transitive automorphism group, and are orientable in even dimensions, non-orientable in odd dimensions. In view of the above results, it follows that the triangulated d -manifolds we construct are tight for $d \geq 2$ and are tight neighborly for $d \geq 3$. Our examples are also strongly minimal. More explicitly we have

Theorem 1.5. *For $d \geq 2$ and $n = d^2 + 5d + 5$, there exist n -vertex members M_n^d and N_n^d of $\mathcal{K}(d)$ with the following properties.*

- (a) M_n^d and N_n^d are neighborly for all d .
- (b) M_n^d and N_n^d have vertex-transitive automorphism groups for all d .
- (c) M_n^d and N_n^d are tight neighborly for $d \geq 3$.
- (d) M_n^d and N_n^d are tight for all d .
- (e) M_n^d and N_n^d are strongly minimal for all d .
- (f) $\beta_1(M_n^d; \mathbb{Z}_2) = \beta_1(N_n^d; \mathbb{Z}_2) = \binom{n-d-1}{2} / \binom{d+2}{2} = d^2 + 5d + 6$ for $d \geq 3$.
- (g) If $d \geq 2$ is even then M_n^d and N_n^d triangulate $(S^{d-1} \times S^1)^{\# \beta}$ and if $d \geq 3$ is odd then M_n^d and N_n^d triangulate $(S^{d-1} \times S^1)^{\# \beta}$, where $\beta = d^2 + 5d + 6$.

For $d \geq 3$, apart from the $(d+2)$ -vertex standard spheres S_{d+2}^d , the Kühnel's complexes K_{2d+3}^d and few sporadic examples (in [2, 6, 15]), our examples $M_{d^2+5d+5}^d$ and $N_{d^2+5d+5}^d$ are the only known tight neighborly triangulated manifolds (cf. Table 1 in Section 6).

2 Preliminaries

All simplicial complexes considered here are finite and abstract. By a triangulated manifold/sphere/ball, we mean an abstract simplicial complex whose geometric carrier is a topological manifold/sphere/ball. We identify two complexes if they are isomorphic.

A d -dimensional simplicial complex is called *pure* if all its maximal faces (called *facets*) are d -dimensional. A d -dimensional pure simplicial complex is said to be a *weak pseudomanifold* if each of its $(d-1)$ -faces is in at most two facets. For a d -dimensional weak pseudomanifold X , the *boundary* ∂X of X is the pure subcomplex of X whose facets are those $(d-1)$ -dimensional faces of X which are contained in unique facets of X . The *dual graph* $\Lambda(X)$ of a pure simplicial complex X is the graph whose vertices are the facets of X , where two facets are adjacent in $\Lambda(X)$ if they intersect in a face of codimension one. A *pseudomanifold* is a weak pseudomanifold with a connected dual graph. All connected triangulated manifolds are automatically pseudomanifolds.

If X is a d -dimensional simplicial complex then, for $0 \leq j \leq d$, the number of its j -faces is denoted by $f_j = f_j(X)$. The vector (f_0, \dots, f_d) is called the *face vector* of X and the number $\chi(X) := \sum_{i=0}^d (-1)^i f_i$ is called the *Euler characteristic* of X . As is well known, $\chi(X)$ is a topological invariant, i.e., it depends only on the homeomorphic type of $|X|$ and, for any field \mathbb{F} , $\chi(X) = \sum_{i=0}^d (-1)^i \beta_i(X; \mathbb{F})$, where $\beta_i(X; \mathbb{F}) = \dim_{\mathbb{F}}(H_i(X; \mathbb{F}))$ is the i -th *Betti number* of X with respect to the field \mathbb{F} . A simplicial complex X is said to be *l -neighbourly* if any l vertices of X form a face of X . In this paper, by a *neighborly* complex, we shall mean a 2-neighborly complex.

A *standard d -ball* is a pure d -dimensional simplicial complex with one facet. The standard ball with facet σ is denoted by $\bar{\sigma}$. A *standard d -sphere* is a simplicial complex isomorphic to the boundary complex of a standard $(d+1)$ -ball. The standard d -ball on the vertex-set V is denoted by $S_{d+2}^d(V)$ (or simply by S_{d+2}^d). A simplicial complex X is called a *stacked d -ball* if there exists a sequence B_1, \dots, B_m of pure d -dimensional simplicial complexes such that B_1 is a standard d -ball, $B_m = X$ and, for $2 \leq i \leq m$, $B_i = B_{i-1} \cup \bar{\sigma}_i$ and $B_{i-1} \cap \bar{\sigma}_i = \bar{\tau}_i$, where σ_i is a d -face of B_i and τ_i is a $(d-1)$ -face of σ_i . Clearly, a stacked ball is a triangulated ball and (hence) is a pseudomanifold. A simplicial complex is called a *stacked d -sphere* if it is (isomorphic to) the boundary of a stacked $(d+1)$ -ball.

Analogous to Walkup's class $\mathcal{K}(d)$, let $\bar{\mathcal{K}}(d)$ be the class of all d -dimensional simplicial complexes all whose vertex-links are stacked $(d-1)$ -balls. Clearly, if $N \in \bar{\mathcal{K}}(d)$ then N is a triangulated manifold with boundary and satisfies

$$\text{skel}_{d-2}(N) = \text{skel}_{d-2}(\partial N). \quad (2)$$

Here $\text{skel}_j(N) = \{\alpha \in N : \dim(\alpha) \leq j\}$ is the j -skeleton of N . From [4, Remark 2.20], it follows:

Proposition 2.1 (Bagchi and Datta). *For $d \geq 4$, the map $M \mapsto \partial M$ is a bijection between $\bar{\mathcal{K}}(d+1)$ and $\mathcal{K}(d)$.*

The following corollary follows from Proposition 2.1 (cf. [6]).

Corollary 2.2. *For $d \geq 4$, if $M \in \bar{\mathcal{K}}(d+1)$ then $\text{Aut}(M) = \text{Aut}(\partial M)$.*

Note that any automorphism φ of a pure simplicial complex X induces an automorphism $\bar{\varphi}$ of the dual graph $\Lambda(X)$ given by $\sigma \mapsto \varphi(\sigma)$ for any facet σ of X . Here we have:

Lemma 2.3. *Let X be a pseudomanifold which is not a cone (i.e., not all the facets are through a single vertex). Then, for any $\varphi \in \text{Aut}(X)$, the induced automorphism $\bar{\varphi}$ on $\Lambda(X)$ is identity if and only if φ is identity.*

Proof. If φ is identity then clearly $\bar{\varphi}$ is identity. Conversely, let φ be such that $\bar{\varphi}$ is identity on $\Lambda(X)$. Thus $\varphi(\sigma) = \sigma$ for each facet σ in X . Let $x \in V(X)$ be arbitrary. Choose facets

α, β such that $x \in \alpha$ and $x \notin \beta$. As $\Lambda(X)$ is connected, there is a path $\alpha_0 \alpha_1 \cdots \alpha_k$ in $\Lambda(X)$ with $\alpha_0 = \alpha$ and $\alpha_k = \beta$. Since $x \in \alpha_0$ and $x \notin \alpha_k$, there exists $l < k$ such that x is in $\alpha_0, \alpha_1, \dots, \alpha_l$ and $x \notin \alpha_{l+1}$. Hence $\alpha_l \setminus \alpha_{l+1} = \{x\}$. Now $\varphi(\alpha_l) = \alpha_l$ and $\varphi(\alpha_{l+1}) = \alpha_{l+1}$ imply $\varphi(x) = x$. Since x was arbitrary, we see that φ is identity on X . \square

A d -dimensional simplicial complex X is called *minimal* if $f_0(X) \leq f_0(Y)$ for every triangulation Y of the geometric carrier $|X|$ of X . We say that X is *strongly minimal* if $f_i(X) \leq f_i(Y)$, $0 \leq i \leq d$, for all such Y . In [3], Bagchi and Datta have shown the following.

Proposition 2.4 (Bagchi and Datta). *For any field \mathbb{F} , each \mathbb{F} -tight member of $\mathcal{K}(d)$ is strongly minimal.*

3 Construction

In this section we present the construction of a neighborly member of $\mathcal{K}(d)$ for every $d \geq 2$. We need some terminology before we proceed.

Given a graph G and a family $\mathcal{T} = \{T_i\}_{i \in \mathcal{I}}$ of induced subtrees of G , we say that $u \in V(G)$ defines the subset $\hat{u} = \{i \in \mathcal{I} : u \in V(T_i)\}$ of \mathcal{I} . Our constructions are based on the following lemma from [6].

Lemma 3.1. *Let G be a graph and $\mathcal{T} = \{T_i\}_{i=1}^n$ be a family of $(n-d)$ -vertex induced subtrees of G , any two of which intersect. Suppose that (i) each vertex of G is in exactly $d+1$ members of \mathcal{T} and (ii) for any two vertices $u \neq v$ of G , u and v are together in exactly d members of \mathcal{T} if and only if uv is an edge of G . Then the pure simplicial complex M whose facets are \hat{u} , where $u \in V(G)$, is a neighborly member of $\bar{\mathcal{K}}(d)$, with $\Lambda(M) \cong G$.*

We consider an example of an intersecting family of induced subtrees of a graph which we use later in our constructions.

Example 3.2. Let $d \geq 2$ and $n = d^2 + 5d + 5$. Consider the graph G^d on $n(d+2)$ vertices consisting of two n -cycles C_1, C_2 and n vertex-independent paths P_i , $0 \leq i \leq n-1$, given by

$$C_1 = \sigma_0 \sigma_1 \cdots \sigma_{n-1} \sigma_0, C_2 = \mu_0 \mu_{d+3} \mu_{2(d+3)} \cdots \mu_{(n-1)(d+3)} \mu_0, P_i = \sigma_i \alpha_{1,i} \alpha_{2,i} \cdots \alpha_{d,i} \mu_i, \quad (3)$$

where the subscripts are modulo n . Let $\mathcal{T} = \{T_i\}_{i=0}^{n-1}$ be the family of induced trees where the vertex-set $V(T_i)$ of T_i is given by (see Figure 1).

$$V(T_i) = \{\sigma_{i+k} : 0 \leq k \leq d+1\} \cup \{\mu_{i+j(d+3)} : 0 \leq j \leq d+1\} \cup \{\alpha_{l,i} : 1 \leq l \leq d\} \cup \left(\bigcup_{k=2}^{d+1} \{\alpha_{l,i+k} : 1 \leq l \leq d+2-k\} \right) \cup \left(\bigcup_{k=2}^{d+1} \{\alpha_{l,i+k(d+3)} : d+2-k \leq l \leq d\} \right). \quad (4)$$

Figure 2 shows the graph G^4 with the tree T_0 in bold.

Lemma 3.3. *For $d \geq 2$, let the graph G^d and the family of induced subtrees $\mathcal{T} = \{T_i\}_{i=0}^{n-1}$ be as defined in (3) and (4) respectively. Then $T_i \cap T_j \neq \emptyset$ for all $0 \leq i, j \leq n-1$.*

Proof. Let φ be a bijection on $V(G^d)$ given by

$$\varphi = (\sigma_0, \dots, \sigma_{n-1})(\mu_0, \dots, \mu_{n-1}) \prod_{j=1}^d (\alpha_{j,0}, \dots, \alpha_{j,n-1}).$$

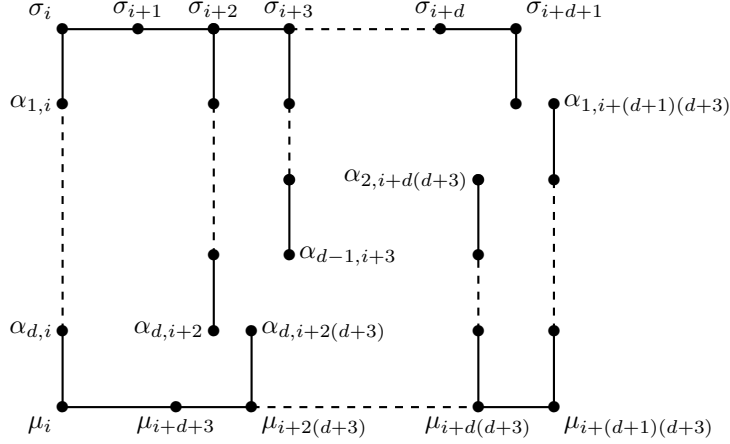


Figure 1: Schematic representation of T_i in G^d

It is easily seen that φ is an automorphism of G^d and further we have $T_{i+1} = \varphi(T_i)$. Thus $T_i = \varphi^i(T_0)$. Thus to show that $T_i \cap T_j \neq \emptyset$ for all $0 \leq i, j \leq n-1$, it is sufficient to show that $T_i \cap T_0 \neq \emptyset$ for $0 \leq i \leq n-1$.

Claim: For $0 \leq i \leq n-1$, if there exist integers l, k with $2 \leq l \leq k \leq d+1$ which satisfy either (i) $i + k(d+3) = n + l$ or (ii) $i + l = k(d+3)$ then T_i intersects T_0 .

Suppose $i + k(d+3) = n + l$ for some integers l, k satisfying $2 \leq l \leq k \leq d+1$. Thus $i + k(d+3) = l \pmod{n}$. Then from (4), we see that $\{\alpha_{j,l} : d+2-k \leq j \leq d\} \subseteq V(T_i)$. Also from (4), $\{\alpha_{j,l} : 1 \leq j \leq d+2-l\} \subseteq V(T_0)$. For $l \leq k$, we see that the intersection of the above two sets is $\{\alpha_{j,l} : d+2-k \leq j \leq d+2-l\} \subseteq V(P_l)$. Next suppose that $i + l = k(d+3)$ for some integers $2 \leq l \leq k \leq d+1$. Again from (4), we have $\{\alpha_{j,i+l} : 1 \leq j \leq d+2-l\} \subseteq V(T_i)$ and $\{\alpha_{j,i+l} : d+2-k \leq j \leq d\} \subseteq V(T_0)$. Thus for $l \leq k$, the two sets intersect, and hence T_0 and T_i intersect. This proves the claim.

Clearly, we have the following six cases.

- (a) $0 \leq i \leq d+1$: In this case, T_i intersects T_0 in σ_i .
- (b) $i > (d+1)(d+3)$: It is easy to see that T_i contains σ_0 and hence $T_i \cap T_0 \neq \emptyset$.
- (c) $i = k(d+3)$, $1 \leq k \leq d+1$: In this case, T_i intersects T_0 in μ_i .
- (d) $i = k(d+3) - 1$, $1 \leq k \leq d+1$: Then $i + l(d+3) = n = 0 \pmod{n}$, where $l = d+2-k \leq d+1$. This implies T_i contains $\mu_0 \in V(T_0)$. So, $T_i \cap T_0 \neq \emptyset$.
- (e) $j(d+3) < i < (j+1)(d+2)$, $1 \leq j \leq d$: Let $i = j(d+3) + t$, where $1 \leq t < d+2-j$. Let $k = d+2-j$. Then $i + k(d+3) = n + l$ where $l = t+1$. Since $1 \leq j \leq d$, we have $2 \leq l \leq k \leq d+1$. Hence, by the claim, T_i intersects T_0 .
- (f) $k(d+2) \leq i < k(d+3) - 1$, $2 \leq k \leq d+1$: Let $i = k(d+2) + t$ where $0 \leq t < k-1$. Let $l = k-t$. Then $i + l = k(d+3)$ and $2 \leq l \leq k \leq d+1$. Therefore, by the claim, T_i intersects T_0 .

This completes the proof of the lemma. \square

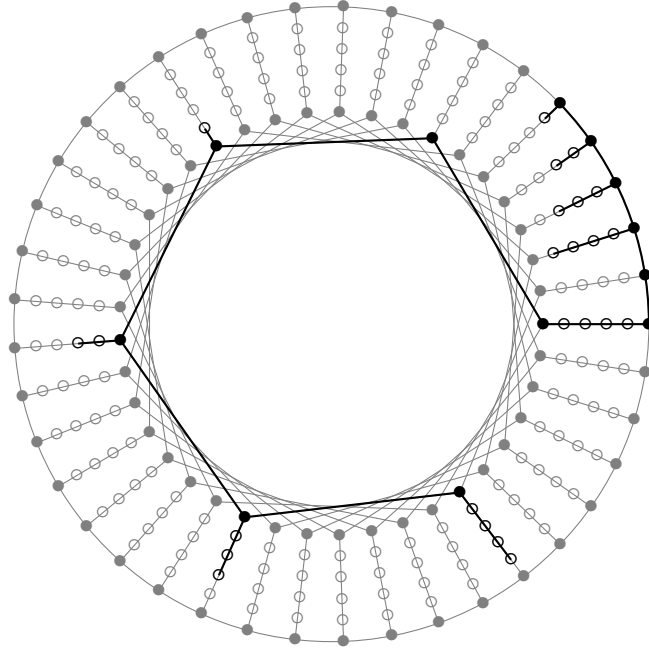


Figure 2: Graph G^4 and the tree T_0

Lemma 3.4. Let G^d be the graph and $\mathcal{T} = \{T_i\}_{i=0}^{n-1}$ be the family of induced subtrees of G^d as defined in (3) and (4) respectively. Then

- (a) T_i is a tree on $n - d - 1$ vertices.
- (b) For all $v \in V(G^d)$, \hat{v} is a $(d + 2)$ -element set.
- (c) For $u, v \in V(G^d)$, $\hat{u} \cap \hat{v}$ is a $(d + 1)$ -element set if and only if uv is an edge in G^d .

Proof. From (4) we have,

$$\#V(T_i) = (d + 2) + (d + 2) + d + \sum_{k=2}^{d+1} (d + 2 - k) + \sum_{k=2}^{d+1} (k - 1) = d^2 + 4d + 4 = n - d - 1.$$

This proves (a). From (4), we see that

$$\begin{aligned} \hat{\alpha}_{l,m} &= \{m\} \cup \{m - k : 2 \leq k \leq d + 2 - l\} \cup \{m - (d + 3)j : d + 2 - l \leq j \leq d + 1\}, \\ \hat{\sigma}_l &= \{l - k : 0 \leq k \leq d + 1\}, \quad \hat{\mu}_l = \{l - k(d + 3) : 0 \leq k \leq d + 1\}. \end{aligned} \quad (5)$$

(Here the elements are modulo n .) Clearly $\hat{\sigma}_l, \hat{\mu}_l$ are sets of size $d + 2$. Further notice that for $2 \leq k, j \leq d + 1$, $k \neq (d + 3)j \pmod{n}$, and hence $\#(\hat{\alpha}_{l,m}) = 1 + (d + 1 - l) + l = d + 2$. This proves (b).

Let us define a metric Δ on the set $V(G^d)$ as $\Delta(u, v) = \#(\hat{u} \setminus \hat{v}) = \#(\hat{v} \setminus \hat{u})$. It is easy to see that Δ indeed defines a metric on $V(G^d)$. Clearly, $\#(\hat{\sigma}_i \cap \hat{\mu}_j) < d + 1$ and $\sigma_i \mu_j$ is not an edge of G^d for $0 \leq i, j \leq n - 1$. Thus, to prove (c), we need to show the following:

- (i) $\Delta(\sigma_i, \sigma_j) = 1 \Leftrightarrow i - j = \pm 1 \pmod{n}$.
- (ii) $\Delta(\mu_i, \mu_j) = 1 \Leftrightarrow i - j = \pm(d + 3) \pmod{n}$.

(iii) $\Delta(\alpha_{l,m}, \alpha_{r,s}) = 1 \Leftrightarrow m = s, l - r = \pm 1$.

(iv) $\Delta(\sigma_i, \alpha_{l,m}) = 1 \Leftrightarrow i = m, l = 1$.

(v) $\Delta(\mu_i, \alpha_{l,m}) = 1 \Leftrightarrow i = m, l = d$.

In all the above cases, the reverse implications follow from the definitions of the sets in (5). Before we proceed with the proofs of the forward implications, we introduce some notations. For integers i, j , let $|i - j|_n$ denote the least non-negative integer k such that either $i + k = j \pmod{n}$, or $j + k = i \pmod{n}$. Note that $|i - j|_n \leq n/2$. For integers $i \leq j$, let $[i, j]_n := \{z \in \mathbb{Z} : z \equiv k \pmod{n}, \text{ for some } k \in \{i, i+1, \dots, j\}\}$.

Claim 1: $\Delta(\sigma_i, \sigma_j) \geq \min\{|i - j|_n, d + 2\}$.

If $|i - j|_n = 0$ then there is nothing to prove. So, assume that $t := |i - j|_n > 0$.

Assume, without loss, that $j = i + t \pmod{n}$. Let $T = \{j - k : 0 \leq k \leq t - 1\}$. We claim that $T \cap \hat{\sigma}_i = \emptyset$. If possible let $T \cap \hat{\sigma}_i \neq \emptyset$. Then there exist integers k, k' , where $0 \leq k \leq t - 1$ and $0 \leq k' \leq d + 1$, such that $j - k = i - k' \pmod{n}$. So, $j = i + (k - k') \pmod{n}$. Since $k - k' \leq t - 1$, this implies (by the definition of $|i - j|_n$) that $k - k' < 0$. Thus, $t - (k - k') > 0$ and $t - (k - k') = 0 \pmod{n}$ (since $t = j - i = k - k' \pmod{n}$). So, $t - (k - k') = pn$ for some positive integer p . Then $n \leq pn \leq t + k' \leq n/2 + (d + 1)$. This implies $n \leq 2d + 2$, a contradiction. Thus, $T \cap \hat{\sigma}_i = \emptyset$.

Now, if $t \leq d + 1$, then $T \subseteq \{j - k : 0 \leq k \leq d + 1\} = \hat{\sigma}_j$ and hence $\Delta(\sigma_i, \sigma_j) \geq \#(T) = t$. On the other hand, if $t \geq d + 2$, then $T \supseteq \{j - k : 0 \leq k \leq d + 1\} = \hat{\sigma}_j$, and hence $\hat{\sigma}_i \cap \hat{\sigma}_j = \emptyset$. Therefore $\Delta(\sigma_i, \sigma_j) = d + 2$. This proves Claim 1.

For $1 \leq i \leq d$, $0 \leq j \leq n - 1$, let $A_{i,j} := \{j\} \cup \{j - k : 2 \leq k \leq d + 2 - i\}$, $B_{i,j} := \{j - k(d + 3) : d + 2 - i \leq k \leq d + 1\}$, $C_{i,j} := \{j\} \cup \{j - k(d + 3) : d + 2 - i \leq k \leq d + 1\}$ and $D_{i,j} := \{j - k : 2 \leq k \leq d + 2 - i\}$. So, $\hat{\alpha}_{i,j} = A_{i,j} \sqcup B_{i,j} = C_{i,j} \sqcup D_{i,j}$.

Claim 2: (a) If $|m - s|_n > d + 1$ then $A_{l,m} \cap A_{r,s} = \emptyset$ and $\#(A_{l,m} \cap B_{r,s}) \leq 1$ for $1 \leq l, r \leq d$.
(b) If $0 < |m - s|_n \leq d + 1$ then $C_{l,m} \cap C_{r,s} = \emptyset$ and $\#(C_{l,m} \cap D_{r,s}) \leq 1$ for $1 \leq l, r \leq d$.

Suppose $|m - s|_n > d + 1$. If possible let $z \in A_{l,m} \cap A_{r,s}$. Then there exist integers k, k' with $0 \leq k \leq d + 2 - l \leq d + 1$ and $0 \leq k' \leq d + 2 - r \leq d + 1$ such that $m - k = z = s - k' \pmod{n}$. Then $|m - s|_n \leq d + 1$, a contradiction. Thus $A_{l,m} \cap A_{r,s} = \emptyset$.

If possible let $z, x \in A_{l,m} \cap B_{r,s}$, where $z \neq x$. Since $z, x \in A_{l,m}$, there exist $a, b \in \{0, \dots, d + 1\}$ such that $z = m - a$ and $x = m - b$. Then $z - x = b - a \in [-(d + 1), (d + 1)]_n$. Since $z, x \in B_{r,s}$, there exist $k, k' \in \{2, \dots, d + 1\}$ such that $z = s - k(d + 3) \pmod{n}$ and $x = s - k'(d + 3) \pmod{n}$. So, $z - x = (k' - k)(d + 3) \pmod{n}$. Assume without loss that $k' > k$. Then $1 \leq k' - k \leq d - 1$ and hence $d + 1 < (k' - k)(d + 3) < n - (d + 1)$. This implies that $z - x = (k' - k)(d + 3) \notin [-(d + 1), (d + 1)]_n$, a contradiction. Therefore, $\#(A_{l,m} \cap B_{r,s}) \leq 1$. This proves part (a). By similar arguments, part (b) of Claim 2 follows.

Claim 3: If $\Delta(\alpha_{l,m}, \alpha_{r,s}) = 1$ then $m = s$.

Assume that $\Delta(\alpha_{l,m}, \alpha_{r,s}) = 1$. Then $\#(\hat{\alpha}_{l,m} \cap \hat{\alpha}_{r,s}) = d + 1$. If possible let $m \neq s$. Then $|m - s|_n > 0$. We have the following two cases.

Case 1. $|m - s|_n > d + 1$. Then, by Claim 2 (a), we have $A_{l,m} \cap A_{r,s} = \emptyset$, $\#(A_{l,m} \cap \hat{\alpha}_{r,s}) \leq 1$ and $\#(A_{r,s} \cap \hat{\alpha}_{l,m}) \leq 1$. Also, $\#(B_{l,m}) = l \leq d$, $\#(B_{r,s}) = r \leq d$ and hence $\#(B_{l,m} \cap B_{r,s}) \leq d$. Since $\#(\hat{\alpha}_{l,m} \cap \hat{\alpha}_{r,s}) = d + 1$, these imply $\#(B_{l,m} \cap B_{r,s}) = d$. This implies $B_{l,m} = B_{r,s}$ and $\#(B_{l,m}) = d = \#(B_{r,s})$. Therefore $l = d = r$. In particular, $B_{d,m} = B_{d,s}$. Then there exist integers $2 \leq k, k' \leq d + 1$ such that $m - 2(d + 3) = s - k(d + 3) \pmod{n}$, and $m - k'(d + 3) = s - 2(d + 3) \pmod{n}$. Subtracting we get $(k' - 2)(d + 3) = (2 - k)(d + 3)$

(mod n). Multiplying by $d+2$, we get $k' - 2 = 2 - k \pmod{n}$ and hence $k + k' = 4 \pmod{n}$. Since $4 \leq k + k' \leq 2d + 2 < n$, it follows that $k = k' = 2$. Thus $m - 2(d+3) = s - 2(d+3) \pmod{n}$ and hence $m = s \pmod{n}$. This is not possible since $0 \leq m, s \leq n-1$ and $m \neq s$.

Case 2. $0 < |m-s|_n \leq d+1$. Then, by Claim 2 (b), we have $C_{l,m} \cap C_{r,s} = \emptyset$, $\#(C_{l,m} \cap \hat{\alpha}_{r,s}) \leq 1$ and $\#(C_{r,s} \cap \hat{\alpha}_{l,m}) \leq 1$. Since $\#(\hat{\alpha}_{l,m} \cap \hat{\alpha}_{r,s}) = d+1$, we must have $\#(D_{l,m} \cap D_{r,s}) = d$. These implies (as in Case 1) $D_{l,m} = D_{r,s}$ and $\#(D_{l,m}) = d = \#(D_{r,s})$. Then, from the definition of $D_{l,m}$ (resp., $D_{r,s}$), $l = r = 1$. So, $D_{1,m} = D_{1,s}$. As in Case 1, we get $m = s \pmod{n}$. Again this is not possible.

Thus, we get contradictions in both the cases. Therefore, $m = s$. This proves Claim 3.

If $\Delta(\sigma_i, \sigma_j) = 1$ then, by Claim 1, $|i - j|_n \leq 1$ and hence $i - j = \pm 1 \pmod{n}$. This proves (i).

Since $(d+2)(d+3) = 1 \pmod{n}$, we see that the map $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ given by $i \mapsto (d+2)i$ is a bijection, with the inverse map π^{-1} given by $i \mapsto (d+3)i$. From the definitions of $\hat{\sigma}_i$ and $\hat{\mu}_i$, we see that $\pi(\hat{\mu}_i) = \hat{\sigma}_{i(d+2)}$. Thus $\Delta(\mu_i, \mu_j) = 1 \Leftrightarrow \Delta(\sigma_{i(d+2)}, \sigma_{j(d+2)}) = 1 \Leftrightarrow |i(d+2) - j(d+2)|_n = 1$. Thus $i(d+2) - j(d+2) = \pm 1 \pmod{n}$, where multiplying by $d+3$ gives $i - j = \pm(d+3) \pmod{n}$. This proves (ii).

Now, assume $\Delta(\alpha_{l,m}, \alpha_{r,s}) = 1$. By Claim 3, $m = s$. So, $\Delta(\alpha_{l,m}, \alpha_{r,m}) = 1$. From the reverse implication, we have $\Delta(u, v) = 1$ whenever uv is an edge in G^d . Notice that $\hat{\sigma}_m \cap \hat{\mu}_m = \{m\}$. Thus $\Delta(\sigma_m, \mu_m) = d+1$. If possible let $l \neq r \pm 1$. Then we can assume, without loss, that $r > l$ and $r - l \geq 2$. Then by triangle inequality we have $d+1 = \Delta(\sigma_m, \mu_m) \leq \Delta(\sigma_m, \alpha_{l,m}) + \Delta(\alpha_{l,m}, \alpha_{r,m}) + \Delta(\alpha_{r,m}, \mu_m) \leq l+1 + (d+1-r) < d+1$, a contradiction. Therefore, $l = r \pm 1$. This completes the proof of (iii).

We can also prove (iv) and (v), by using the triangle inequality for Δ along the path $P_m = \sigma_m \alpha_{1,m} \cdots \alpha_{d,m} \mu_m$. This completes the proof of the lemma. \square

Example 3.5. Let $d \geq 2$ and $n = d^2 + 5d + 5$. Let (G^d, \mathcal{T}) be as in Example 3.2. By Lemma 3.4, $\hat{v} = \{i : v \in T_i\}$ is a set of $d+2$ elements for each $v \in V(G^d)$. Consider the $(d+1)$ -dimensional simplicial complex \mathcal{M}_n^{d+1} consisting of facets \hat{v} , $v \in V(G^d)$. From Lemmas 3.3 and 3.4, we see that (G^d, \mathcal{T}) satisfy the hypothesis of Lemma 3.1 and hence, by Lemma 3.1, \mathcal{M}_n^{d+1} is a neighborly member of $\overline{\mathcal{K}}(d+1)$. For notational convenience we shall denote the vertex-set of \mathcal{M}_n^{d+1} by $\{a_0, a_1, \dots, a_{n-1}\}$ with the identification $i \leftrightarrow a_i$. We also identify \hat{u} with u for $u \in V(G^d)$. Under this identification we can write the facets (see (5)) of \mathcal{M}_n^{d+1} as:

$$\begin{aligned} \sigma_i \equiv \hat{\sigma}_i &= a_{i-d-1} a_{i-d} \cdots a_{i-1} a_i, \\ \mu_i \equiv \hat{\mu}_i &= a_{i+(d+2)} a_{i+2(d+3)-1} \cdots a_{i+(d+1)(d+3)-1} a_i, \\ \alpha_{k,i} \equiv \hat{\alpha}_{k,i} &= a_{i-2-d+k} \cdots a_{i-2} a_i a_{i+(d+2)} a_{i+2(d+3)-1} \cdots a_{i+k(d+3)-1}, \end{aligned} \quad (6)$$

$0 \leq i \leq n-1$, $1 \leq k \leq d$. The subscripts are modulo n . We further define,

$$M_n^d := \partial \mathcal{M}_n^{d+1}. \quad (7)$$

Since $\mathcal{M}_n^{d+1} \in \overline{\mathcal{K}}(d+1)$, we have $M_n^d \in \mathcal{K}(d)$. By (2), $\text{skel}_{d-1}(M_n^d) = \text{skel}_{d-1}(\mathcal{M}_n^{d+1})$. This implies that $f_0(M_n^d) = f_0(\mathcal{M}_n^{d+1}) = n$ and, since $d \geq 2$, M_n^d is neighborly.

Lemma 3.6. Let α be the permutation of order n on $V(M_n^d)$ given by $\alpha(a_i) = a_{i+1}$ (addition in the subscript is modulo n). Then

- (a) α is an automorphism of both \mathcal{M}_n^d and M_n^d .

(b) $\text{Aut}(\mathcal{M}_n^{d+1})$ (resp., $\text{Aut}(M_n^d)$) acts transitively on $V(\mathcal{M}_n^{d+1})$ (resp., on $V(M_n^d)$).

(c) $\text{Aut}(\mathcal{M}_n^{d+1}) = \langle \alpha \rangle \cong \mathbb{Z}_n$.

Proof. Observe that the bijection $\alpha: a_i \mapsto a_{i+1}$ (i.e., $i \mapsto i+1 \pmod n$) induces the following permutation of facets of \mathcal{M}_n^{d+1} .

$$(\sigma_0, \dots, \sigma_{n-1})(\mu_0, \dots, \mu_{n-1}) \prod_{j=1}^d (\alpha_{j,0}, \dots, \alpha_{j,n-1}).$$

Thus α is an automorphism of \mathcal{M}_n^{d+1} . Since any automorphism of \mathcal{M}_n^{d+1} is an automorphism of $\partial\mathcal{M}_n^{d+1}$, it follows that $\alpha \in \text{Aut}(M_n^d)$. This proves (a).

Clearly, $\langle \alpha \rangle$ is transitive on $V(\mathcal{M}_n^{d+1}) = V(M_n^d)$. Part (b) follows from this.

Let $\beta \in \text{Aut}(\mathcal{M}_n^{d+1})$. Let $\bar{\beta} \in \text{Aut}(\Lambda(\mathcal{M}_n^{d+1}))$ be the induced automorphism. If $\beta(a_0) = a_0$ then $\beta(\text{lk}_{\mathcal{M}_n^{d+1}}(a_0)) = \text{lk}_{\mathcal{M}_n^{d+1}}(a_0)$ and hence $\bar{\beta}(T_0) = T_0$ and $\bar{\beta}$ is an automorphism of the tree T_0 . Then $\bar{\beta}(\sigma_i) = \sigma_i$, $\bar{\beta}(\mu_{i(d+3)}) = \mu_{i(d+3)}$ and $\bar{\beta}(\alpha_{j,0}) = \alpha_{j,0}$ for $0 \leq i \leq d+1$, $1 \leq j \leq d$. This implies $\bar{\beta}|_{C_1} = \text{Id}$, $\bar{\beta}|_{C_2} = \text{Id}$. These imply that $\bar{\beta}$ is the identity of $\text{Aut}(\Lambda(\mathcal{M}_n^{d+1}))$. Then, by Lemma 2.3, that β is the identity of $\text{Aut}(\mathcal{M}_n^{d+1})$. Thus the only automorphism of \mathcal{M}_n^{d+1} which fixes a_0 is the identity automorphism. Since $\langle \alpha \rangle$ is transitive on $V(\mathcal{M}_n^{d+1})$, this implies that $\langle \alpha \rangle = \text{Aut}(\mathcal{M}_n^{d+1})$. This proves (c). \square

4 Another series of neighborly members of $\mathcal{K}(d)$

A slight modification in the choice of graph G^d and the induced trees $\mathcal{T} = \{T_i\}_{i=0}^{n-1}$, gives us another series of tight triangulations (as before $n = d^2 + 5d + 5$).

Example 4.1. Let $d \geq 2$ and $n = d^2 + 5d + 5$. Consider the graph \tilde{G}^d on $n(d+2)$ vertices consisting of two n -cycles \tilde{C}_1 , \tilde{C}_2 and n vertex-independent paths \tilde{P}_i , $0 \leq i \leq n-1$, given by

$$\tilde{C}_1 = \sigma_0 \sigma_1 \cdots \sigma_{n-1} \sigma_0, \tilde{C}_2 = \mu_0 \mu_{d+2} \mu_{2(d+2)} \cdots \mu_{(n-1)(d+2)} \mu_0, \tilde{P}_i = \sigma_i \alpha_{1,i} \alpha_{2,i} \cdots \alpha_{d,i} \mu_i, \quad (8)$$

where the subscripts are modulo n . Let $\tilde{\mathcal{T}} = \{\tilde{T}_i\}_{i=0}^{n-1}$ be the family of induced trees where the vertex-set $V(\tilde{T}_i)$ of \tilde{T}_i is given by (see Figure 3).

$$V(\tilde{T}_i) = \{\sigma_{i+j} : 0 \leq j \leq d+1\} \cup \{\mu_{i+j(d+2)} : 0 \leq j \leq d+1\} \cup \{\alpha_{j,i} : 1 \leq j \leq d\} \cup \\ \left(\bigcup_{k=2}^{d+1} \{\alpha_{j,i+k} : 1 \leq j \leq k-1\} \right) \cup \left(\bigcup_{k=2}^{d+1} \{\alpha_{j,i+k(d+2)} : d+2-k \leq j \leq d\} \right). \quad (9)$$

Example 4.2. Let $d \geq 2$ and $n = d^2 + 5d + 5$. Let $(\tilde{G}^d, \tilde{\mathcal{T}})$ be as in Example 4.1. It can be shown (as in Section 3), that \tilde{G}^d and $\tilde{\mathcal{T}}$ satisfy the hypothesis of Lemma 3.1 and hence, by Lemma 3.1, yield a neighborly member of $\overline{\mathcal{K}}(d+1)$, which we denote by \mathcal{N}_n^{d+1} . As before, we take the vertex-set of \mathcal{N}_n^{d+1} is $\{a_0, \dots, a_{n-1}\}$. Thus the facets of \mathcal{N}_n^{d+1} are $v \equiv \hat{v} := \{a_i : v \in V(\tilde{T}_i)\}$, $v \in V(\tilde{G}^d)$. More explicitly, the facets of \mathcal{N}_n^{d+1} are

$$\begin{aligned} \sigma_i &= a_{i-d-1} a_{i-d} \cdots a_{i-1} a_i, \\ \mu_i &= a_{i-(d+1)(d+2)} a_{i-d(d+2)} \cdots a_{i-(d+2)} a_i, \\ \alpha_{k,i} &= a_{i-(d+1)(d+2)} \cdots a_{i-(d+2-k)(d+2)} a_{i-d-1} \cdots a_{i-k-1} a_i, \end{aligned} \quad (10)$$

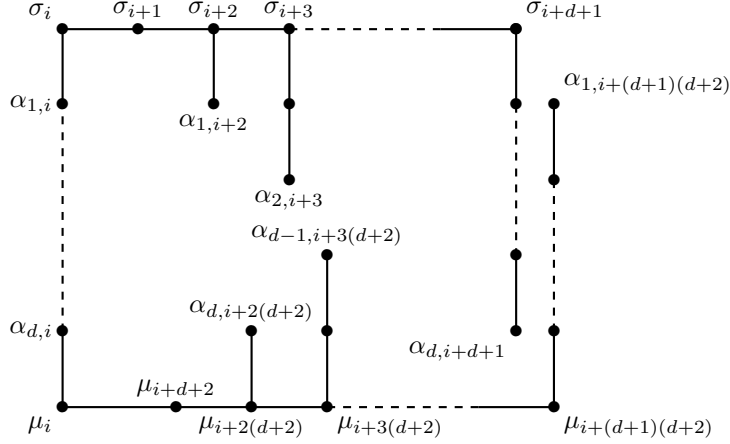


Figure 3: Schematic representation of \tilde{T}_i in \tilde{G}^d

$0 \leq i \leq n-1$, $1 \leq k \leq d$. The subscripts are modulo n . We further define,

$$N_n^d := \partial \mathcal{N}_n^{d+1}. \quad (11)$$

Since $\mathcal{N}_n^{d+1} \in \overline{\mathcal{K}}(d+1)$, we have $N_n^d \in \mathcal{K}(d)$. By the similar arguments as in the case of M_n^d , N_n^d has n vertices and is neighborly.

Lemma 4.3. *Let α be the permutation of order n on $V(N_n^d)$ given by $\alpha(a_i) = a_{i+1}$. Then*

- (a) α is an automorphism of both \mathcal{N}_n^d and N_n^d .
- (b) $\text{Aut}(\mathcal{N}_n^{d+1})$ (resp., $\text{Aut}(N_n^d)$) acts transitively on $V(\mathcal{N}_n^{d+1})$ (resp., on $V(N_n^d)$).
- (c) $\text{Aut}(\mathcal{N}_n^{d+1}) = \langle \alpha \rangle \cong \mathbb{Z}_n$.

Proof. The permutation α induces the following permutation of facets of \mathcal{N}_n^{d+1} .

$$(\sigma_0, \dots, \sigma_{n-1})(\mu_0, \dots, \mu_{n-1}) \prod_{j=1}^d (\alpha_{j,0}, \dots, \alpha_{j,n-1}).$$

Thus $\alpha \in \text{Aut}(\mathcal{N}_n^{d+1}) \subseteq \text{Aut}(\partial \mathcal{N}_n^{d+1})$. This proves (a).

Clearly, $\langle \alpha \rangle$ is transitive on $V(\mathcal{N}_n^{d+1}) = V(N_n^d)$. Part (b) follows from this.

Let $\beta \in \text{Aut}(\mathcal{N}_n^{d+1})$. Let $\bar{\beta} \in \text{Aut}(\Lambda(\mathcal{N}_n^{d+1}))$ be the induced automorphism. Then $\bar{\beta}(T_i) = T_j$, where $\beta(a_i) = a_j$, for all i . If $\beta(a_0) = a_0$ then $\beta(\text{lk}_{\mathcal{N}_n^{d+1}}(a_0)) = \text{lk}_{\mathcal{N}_n^{d+1}}(a_0)$ and hence $\bar{\beta}(T_0) = T_0$ and $\bar{\beta}$ is an automorphism of the tree T_0 . This implies that $\bar{\beta}(\{\sigma_1, \dots, \sigma_{d+1}, \mu_{d+2}, \dots, \mu_{(d+1)(d+2)}\}) = \{\sigma_1, \dots, \sigma_{d+1}, \mu_{d+2}, \dots, \mu_{(d+1)(d+2)}\}$. Since T_0 and T_1 are the only trees which contain $\{\sigma_1, \dots, \sigma_{d+1}, \mu_{d+2}, \dots, \mu_{(d+1)(d+2)}\}$, it follows that $\bar{\beta}(T_1) = T_1$. Inductively, we get $\bar{\beta}(T_i) = T_i$ for all i . Since $\bar{\beta}$ is an automorphism of $\Lambda(\mathcal{N}_n^{d+1})$, β is an automorphism of T_i for all i . This implies that $\bar{\beta}(\alpha_{\lfloor d/2 \rfloor, i}) = \alpha_{\lfloor d/2 \rfloor, i}$ or $\alpha_{\lceil d/2 \rceil, i}$ for all i . Now, either $\bar{\beta}$ is identity on T_0 or $\bar{\beta}(\sigma_j) = \mu_{j(d+2)}$ for $0 \leq j \leq d+1$. In the second case, $\bar{\beta}(\alpha_{\lfloor d/2 \rfloor, d+1}) = \alpha_{\lceil d/2 \rceil, (d+1)(d+2)}$. This is not possible since $\bar{\beta}(\alpha_{\lfloor d/2 \rfloor, d+1}) = \alpha_{\lfloor d/2 \rfloor, d+1}$ or $\alpha_{\lceil d/2 \rceil, d+1}$. Therefore, $\bar{\beta}|_{T_0}$ is the identity. Now, by the similar argument as in the proof of Lemma 3.6 it follows that β is the identity in $\text{Aut}(\mathcal{N}_n^{d+1})$ and $\text{Aut}(\mathcal{N}_n^{d+1}) = \langle \alpha \rangle$. This proves (c). \square

Corollary 4.4. *For $d \geq 2$ and $n = d^2 + 5d + 5$, $\text{Aut}(M_n^d) \cong \mathbb{Z}_n \cong \text{Aut}(N_n^d)$.*

Proof. If $d \geq 4$ then the result follows from Lemmas 3.6 (c), 4.3 (c) and Corollary 2.2.

From Lemmas 3.6 and 4.3, it follows that $\text{Aut}(M_n^d)$ (resp., $\text{Aut}(N_n^d)$) has a subgroup isomorphic to \mathbb{Z}_n for $2 \leq d \leq 3$ also. Using simpcomp [9], we found that $\text{Aut}(M_{19}^2) \cong \mathbb{Z}_{19} \cong \text{Aut}(N_{19}^2)$ and $\text{Aut}(M_{29}^3) \cong \mathbb{Z}_{29} \cong \text{Aut}(N_{29}^3)$. This completes the proof. \square

5 Proof of Theorem 1.5

For $d \geq 2$ and $m \geq 2$, let D_{m+d+1}^{d+1} be the stacked $(d+1)$ -ball with vertex-set $\{1, 2, \dots, m+d+1\}$ and facets $\{k, k+1, \dots, k+d+1\}$, $1 \leq k \leq m$. Let $M = \partial D_{m+d+1}^{d+1}$, A be the d -simplex $\{1, 2, \dots, d+1\}$ and B be the d -simplex $\{m+1, m+2, \dots, m+d+1\}$. Then $M' := M \setminus \{A, B\}$ triangulates $I \times S^{d-1}$. Recall that, a bijection $\psi: A \rightarrow B$ is called *admissible* if for each vertex $u \in A$, there does not exist $v \in V(M)$ such that both $\{u, v\}$ and $\{\psi(u), v\}$ are edges in M ([1]). If $m \leq 2d+2$ then $\{d+1, d+1+\lceil \frac{m}{2} \rceil\}$ and $\{d+1+\lceil \frac{m}{2} \rceil, j\}$ are edges in M for $d+1+\lceil \frac{m}{2} \rceil \neq j \in B$. Thus, existence of an admissible map implies that $m \geq 2d+3$. On the other hand, if $m \geq 3d+3$ then there is no common neighbour of i and j in M for $i \in A$, $j \in B$ and hence any bijection $\psi: A \rightarrow B$ is admissible. Let σ be a permutation on the set $\{1, \dots, d+1\}$ (i.e., $\sigma \in \text{Sym}(d+1)$). Consider the bijection $\varphi_\sigma: A \rightarrow B$ given by $\varphi_\sigma(i) = m + \sigma(i)$. Consider the quotient complexes $Y := D_{m+d+1}^{d+1}/\varphi_\sigma$ and $X_m^d(\sigma) := M'/\varphi_\sigma$. Then $\partial Y = X_m^d(\sigma)$. If φ_σ is admissible then $X_m^d(\sigma) \in \mathcal{K}(d)$ and triangulates an S^{d-1} -bundle over S^1 . The case when md is even of the following lemma was proved in Lemma 3.3 of [1].

Lemma 5.1. *For $d \geq 2$, let $X_m^d(\sigma)$ be as above, where φ_σ is admissible. Then $X_m^d(\sigma)$ is orientable if and only if either md is even and σ is an even permutation or md is odd and σ is an odd permutation. (In particular, $X_m^d(\text{Id})$ is orientable when md is even and non-orientable when md is odd.)*

Proof. For $1 \leq k \leq m$, $1 \leq l \leq d$, let $\delta_{k,l}$ denote the d -simplex $\{k, k+1, \dots, k+d+1\} \setminus \{k+l\}$ of M . Since $|M'|$ is homeomorphic to $[0, 1] \times |\partial B|$, M' is orientable. Observe that the following defines an orientation on $|M'|$. (Here $\partial B = S_{d+1}^{d-1}(B)$.)

$$+ \delta_{k,l} = (-1)^{kd+l+1} \langle k, k+1, \dots, k+l-1, k+l+1, \dots, k+d+1 \rangle. \quad (12)$$

(To check that (12) defines a coherent orientation, one can take any orientation on $(d-1)$ -simplices of M' . In particular, one can take positively oriented $(d-1)$ -simplices as given in (13) below.)

We can choose an orientation on $|\partial B|$ so that the orientation on $|M'|$ as the product $[0, 1] \times |\partial B|$ is the same as the orientation given in (12). This also induces an orientation on $|\partial A|$. Let S_B (resp., S_A) denote the oriented sphere $|\partial B|$ (resp., $|\partial A|$) with this orientation. Now, as the boundary of an oriented manifold, $\partial|M'| = S_A \cup (-S_B)$ (cf. [7, pages 371–372]). Therefore, $|M'|/\varphi_\sigma = |M'|/|\varphi_\sigma|$ is orientable if and only if $|\varphi_\sigma|: S_A \rightarrow S_B$ is orientation preserving (cf. [16, pages 134–135]). (Here, $|\varphi_\sigma|: |\partial A| \rightarrow |\partial B|$ is the homeomorphism induced by φ_σ .)

Note that $(d-1)$ -simplices of M are $\delta_{k,i,j} = \{k, k+1, \dots, k+d+1\} \setminus \{k+i, k+j\}$, $0 \leq i < j \leq d+1$, $(i, j) \neq (0, d+1)$, $1 \leq k \leq m$. Consider the orientation on the $(d-1)$ -skeleton of M' as:

$$+ \delta_{k,i,j} = (-1)^{kd+i+j} \langle k, \dots, k+i-1, k+i+1, \dots, k+j-1, k+j+1, \dots \rangle. \quad (13)$$

Then $[\delta_{m,i+1}, \delta_{m,0,i+1}] = -1$ (resp., $[\delta_{1,i}, \delta_{1,i,d+1}] = 1$) for $0 \leq i \leq d$. This implies that $|\partial B|$ (resp., $|\partial A|$) with orientation given in (13) is S_B (resp., S_A). (For $\beta = \{v_0, v_1, \dots, v_d\} \in M'$ and $\alpha = \{v_1, \dots, v_d\} \in \partial B$, if $[\beta, \alpha] = -1$ then $+\alpha = \langle v_1, \dots, v_d \rangle$ with the orientation given in (13) $\iff +\beta = \langle v_1, v_0, v_2, \dots, v_d \rangle$ with the orientation given in (12) $\iff (\overrightarrow{v_1 v_0}, \overrightarrow{v_1 v_2}, \dots, \overrightarrow{v_1 v_d})$ is the orientation of $|M'| \iff (\overrightarrow{v_1 v_2}, \dots, \overrightarrow{v_1 v_d})$ is the orientation of $|\partial B| \iff \langle v_1, v_2, \dots, v_d \rangle$ is positive in S_B .)

For $1 \leq i \leq d+1$, consider the $(d-1)$ -simplex $\delta_{1,i,d+1} = \{1, \dots, d+1\} \setminus \{i+1\}$ of ∂A . Then $\varphi_{\text{Id}}(\delta_{1,i,d+1}) = \{m+1, \dots, m+d+1\} \setminus \{m+i+1\} = \delta_{m,0,i+1}$. Therefore, from (13), $\varphi_{\text{Id}}(+\delta_{1,i,d+1}) = (-1)^{md} \delta_{d,0,i+1}$. Thus, $|\varphi_{\text{Id}}|: S_A \rightarrow S_B$ is orientation preserving (resp., reversing) if md is even (resp., odd). Also $|\sigma|: S_A \rightarrow S_A$ is orientation preserving (resp., reversing) if σ is an even (resp., odd) permutation. Since $\varphi_\sigma = \varphi_{\text{Id}} \circ \sigma$, it follows that $|\varphi_\sigma|: S_A \rightarrow S_B$ is orientation preserving if and only if md is even and σ is an even permutation or if md is odd and σ is an odd permutation. The lemma now follows. \square

Lemma 5.2. *For $d \geq 2$ and $n = d^2 + 5d + 5$, let M_n^d and N_n^d be as in Examples 3.5 and 4.2 respectively. Then M_n^d, N_n^d are orientable if d is even and are non-orientable if d is odd.*

Proof. We present a proof for M_n^d . Similar arguments work for N_n^d . Let \mathcal{M}_n^{d+1} be as in Example 3.5. Let E_1 (resp., E_2) be the pure $(d+1)$ -dimensional subcomplex of \mathcal{M}_n^{d+1} whose facets are $\sigma_0, \dots, \sigma_{n-1}$ (resp., μ_0, \dots, μ_{n-1}). So, $\Lambda(E_i) = C_i$, $1 \leq i \leq 2$.

Clearly, E_1 is isomorphic to the pseudomanifold $D_{n+d+1}^{d+1}/\varphi_{\text{Id}}$, where D_{n+d+1}^{d+1} is the stacked $(d+1)$ -ball defined at the beginning of this section. Thus, ∂E_1 is isomorphic to $X_n^d(\text{Id})$. Therefore, ∂E_1 triangulates an S^{d-1} -bundle over S^1 and, by Lemma 5.1, is orientable if and only if dn is even. Thus (since n is odd), ∂E_1 is orientable if and only if d is even. So, if d is odd then ∂E_1 is non-orientable and hence (since $|M_n^d|$ can be obtained from $|\partial E_1|$ by attaching handles) M_n^d is non-orientable.

Again, the bijection $f: \mathbb{Z}_n \rightarrow V(E_2)$ given by $f(i) = a_{(d+3)i}$ defines an isomorphism between $D_{n+d+1}^{d+1}/\varphi_{\text{Id}}$ and E_2 . Thus, ∂E_2 is isomorphic to $X_n^d(\text{Id})$. Therefore, ∂E_2 triangulates an S^{d-1} -bundle over S^1 and, by Lemma 5.1, orientable if and only if d is even.

For $0 \leq i \leq n-1$, let F_i be the stacked $(d+1)$ -ball whose facets are $\alpha_{1,i}, \dots, \alpha_{d,i}$. Then, $\mathcal{M}_n^{d+1} = E_1 \cup E_2 \cup (\cup_{i=0}^{n-1} F_i)$ and M_n^d is obtained from $\partial E_1 \cup \partial E_2$ by attaching handles $\partial F_i \setminus \{A_i, B_i\}$, where $A_i = a_{i-d-1} \cdots a_{i-2} a_i$ and $B_i = a_{i+d+2} a_{i+2(d+3)-1} \cdots a_{i+d(d+3)-1} a_i$, $0 \leq i \leq n-1$ (additions are modulo n).

Now, assume that d is even. So, $\partial E_1, \partial E_2, \partial F_i$, $0 \leq i \leq n-1$, are orientable. Consider the orientation on $\partial E_1, \partial E_2$ and ∂F_i , $0 \leq i \leq n-1$, given by:

$$+\sigma_{i,l} = (-1)^l \langle a_{i-d-1}, \dots, a_{i-d-2+l}, a_{i-d+l}, \dots, a_i \rangle, \quad (14)$$

$$+\mu_{i,l} = (-1)^{l+1} \langle a_{i+(d+2)}, \dots, a_{i+l(d+3)-1}, a_{i+(l+2)(d+3)-1}, \dots, a_{i+(d+1)(d+3)-1}, a_i \rangle, \quad (15)$$

$$+\alpha_{k,i,l} = (-1)^l \langle b_{k,i,1}, \dots, b_{k,i,l}, b_{k,i,l+2}, \dots, b_{k,i,d+2} \rangle, \quad (16)$$

where $(b_{k,i,1}, b_{k,i,2}, \dots, b_{k,i,d+2}) = (a_{i-2-d+k}, \dots, a_{i-2} a_i a_{i+(d+2)} a_{i+2(d+3)-1}, \dots, a_{i+k(d+3)-1})$, for $1 \leq k \leq d$, $0 \leq l \leq d+1$. From the proof of Lemma 5.1, (14) (resp., (15)) defines an orientation on ∂E_1 (resp., ∂E_2). Also, (16) defines an orientation on ∂F_i , $0 \leq i \leq n-1$.

Observe that $A_i = \sigma_{i,d} = \alpha_{1,i,d+1}$ and $+\sigma_{i,d} = -\alpha_{1,i,d+1}$. Also, $B_i = \mu_{i,d} = \alpha_{d,i,0}$ and $+\mu_{i,d} = -\alpha_{d,i,0}$. Now, let γ be a $(d-1)$ -face of A_i . Let γ_{E_1} (resp., γ_{F_i}) be the d -face of ∂E_1 (resp., ∂F_i) other than A_i which contains γ . Then (with any orientation of γ)

$$[\gamma_{E_1}, \gamma] = -[\sigma_{i,d}, \gamma] = [\alpha_{1,i,d+1}, \gamma] = -[\gamma_{F_i}, \gamma]. \quad (17)$$

Similarly, if β is a $(d-1)$ -face of B_i and β_{E_2} (resp., β_{F_i}) is the d -face of ∂E_2 (resp., ∂F_i) other than B_i which contains β . Then (with any orientation of β)

$$[\beta_{E_2}, \beta] = -[\mu_{i,d}, \beta] = [\alpha_{d,i,0}, \beta] = -[\beta_{F_i}, \beta]. \quad (18)$$

Since

$$M_n^d = (\partial E_1 \setminus \{A_0, \dots, A_{n-1}\}) \cup (\partial E_2 \setminus \{B_0, \dots, B_{n-1}\}) \cup \left(\bigcup_{i=0}^{n-1} (\partial F_i \setminus \{A_i, B_i\}) \right),$$

it follows from (17) and (18) that the orientations defined by (14), (15) and (16) give a coherent orientation on M_n^d . Thus, M_n^d is orientable. This completes the proof. \square

We need the $d = 3$ case of the following lemma to prove our main result.

Lemma 5.3. *For any field \mathbb{F} and $d \geq 3$, $\beta_1(M_{d^2+5d+5}^d; \mathbb{F}) = \beta_1(N_{d^2+5d+5}^d; \mathbb{F}) = d^2 + 5d + 6$.*

Proof. Let $n = d^2 + 5d + 5$. Let E_1, E_2, F_i, A_i, B_i be as in the proof of Lemma 5.2. Let $X_0 = E_1 \cup E_2$ and $X_i = E_1 \cup E_2 \cup F_0 \cup \dots \cup F_{i-1}$ for $1 \leq i \leq n$. So, $X_n = \mathcal{M}_n^{d+1}$.

Observe that $|E_1|$ (resp., $|E_2|$) is a disc-bundle over circle and only simplices common between E_1 and E_2 are n vertices. This implies that $|X_0|$ is homotopic to union of two circles with n common points. Therefore, $H_1(|X_0|; \mathbb{F}) \cong \mathbb{F}^{n+1}$. Now, for $1 \leq i \leq n$, $|X_i|$ is the union of $|X_{i-1}|$ and the (stacked) ball $|F_{i-1}|$, and $|X_{i-1}| \cap |F_{i-1}| = |A_{i-1}| \cup |B_{i-1}|$. Since $|A_{i-1}| \cup |B_{i-1}|$ is the union of two d -balls with one common point (namely, the vertex a_{i-1}), the j -th reduced homology $H_j^\#(|A_{i-1}| \cup |B_{i-1}|; \mathbb{F}) = \{0\}$ for all j . Therefore, by Mayer-Vietoris sequence, $H_1(|X_i|; \mathbb{F}) \cong H_1(|X_{i-1}|; \mathbb{F})$. This implies that $H_1(|X_n|; \mathbb{F}) \cong H_1(|X_0|; \mathbb{F}) \cong \mathbb{F}^{n+1}$. Thus, $H_1(\mathcal{M}_n^{d+1}; \mathbb{F}) \cong \mathbb{F}^{n+1}$. Since $\text{skel}_{d-1}(\mathcal{M}_n^{d+1}) = \text{skel}_{d-1}(M_n^d)$ (cf. (2)) it follows that $H_j(M_n^d; \mathbb{F}) \cong H_j(\mathcal{M}_n^{d+1}; \mathbb{F})$ for $j \leq d-2$. Since $d \geq 3$, $H_1(M_n^d; \mathbb{F}) \cong H_1(\mathcal{M}_n^{d+1}; \mathbb{F})$ and hence $\beta_1(M_n^d; \mathbb{F}) = n + 1$. Similarly, $\beta_1(N_n^d; \mathbb{F}) = n + 1$. \square

Proof of Theorem 1.5. Part (a) follows from the definitions of M_n^d and N_n^d . Part (b) follows from Lemmas 3.6 (b) and 4.3 (b). Part (e) follows from part (d) and Proposition 2.4.

For the proofs of parts (c), (d), (f) and (g), we first assume that $d \geq 4$. In this case, (c) and (f) follows from Proposition 1.4. Now, (g) follows from Proposition 1.1 and Lemma 5.2. Part (d) follows Proposition 1.2. These prove the theorem for $d \geq 4$.

Next consider the case for $d = 2$. Again, (d) follows from Proposition 1.2. Since M_{19}^2 and N_{19}^2 are orientable (by Lemma 5.2) and neighborly, $\beta_1(M_{19}^2; \mathbb{Z}) = \beta_1(N_{19}^2; \mathbb{Z}) = 2 - (19 - \binom{19}{2} + \frac{2}{3}\binom{19}{2}) = 40$ and hence M_{19}^2 and N_{19}^2 both triangulate $(S^1 \times S^1)^{\#20}$. This proves (g) in this case. So, the theorem holds for $d = 2$.

Finally, assume that $d = 3$. Part (f) follows from Lemma 5.3. Part (c) follows from part (f) and Proposition 1.4 and part (d) follows from part (f) and Proposition 1.3. Part (g) now follows from part (f), Lemma 5.2 and Proposition 1.1. This completes the proof. \square

6 Summary: Known neighborly members of $\mathcal{K}(d)$

Any neighborly triangulated 2-manifold is a neighborly member of $\mathcal{K}(2)$. In Table 1, we summarize the known and some open cases for neighborly members of Walkup's class $\mathcal{K}(d)$ for $d \geq 3$.

$\beta_1(K)$	d	n	K	$ K $	References
0	d	$d + 2$	S_{d+2}^d	S^d	
1	d even	$2d + 3$	K_{2d+3}^d	$S^{d-1} \times S^1$	[11]
1	d odd	$2d + 3$	K_{2d+3}^d	$S^{d-1} \times S^1$	[11]
2	$d \geq 4$	—	Not possible		[14]
3	4	15	M_{15}^4	$(S^3 \times S^1)^{\#3}$	[2]
3	4	15	N_{15}^4	$(S^3 \times S^1)^{\#3}$	[15]
5	5	21	?		[8]
7	4	20	?		[12]
8	4	21	M_{21}^4	$(S^3 \times S^1)^{\#8}$	[6]
8	4	21	N_{21}^4	$(S^3 \times S^1)^{\#8}$	[6]
14	4	26	N_{26}^4	$(S^3 \times S^1)^{\#14}$	[6]
$d^2 + 5d + 6$	d even	$d^2 + 5d + 5$	$M_{d^2+5d+5}^d$	$(S^{d-1} \times S^1)^{\# \beta_1}$	this paper
$d^2 + 5d + 6$	d odd	$d^2 + 5d + 5$	$M_{d^2+5d+5}^d$	$(S^{d-1} \times S^1)^{\# \beta_1}$	this paper

Table 1: Known and some open cases for neighborly members of $\mathcal{K}(d)$, $d \geq 3$

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